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RIEMANNIAN AND PSEUDO-RIEMANNIAN VECTOR BUNDLES WITH EXTRA STRUCTURE

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Abstract

Some basic properties of Riemannian and pseudo-Riemannian vector bundles are developed in this paper. By introducing the notions of vector bundle $\xi = (E, \pi, B, F)$, Riemannian vector bundle (ξ, g), Riemannian coordinate representation $\{(U_\alpha, \varphi_\alpha)\}$, we establish some significant results related to these concepts. Every vector bundle ξ admits a Riemannian metric. If $\xi = (E, \pi, B, F)$ is a vector bundle with Riemannian metric g and $\langle , \rangle$ is a fixed Euclidean inner product in F, then there is a coordinate representation $\{(U_\alpha, \varphi_\alpha)\}$ for ξ such that the maps $\varphi_{\alpha,x} : F \to F_x$ are isometries. Finally, oriented Riemannian bundle and complex vector bundle are defined and it is shown that if $\eta = (E, \pi, B, F_{\mathbb{R}})$ is a real vector bundle of rank $2r$ and $\gamma \in L_\eta$ be a strong bundle map such that $\gamma^2 = -\iota$, then η is the underlying real vector bundle of a complex bundle $\xi = (E, \pi, B, F)$ with complex structure γ.

***Keywords*:** Vector bundle, Riemannian metric, pseudo-Riemannian metric, bundle map.

Introduction

The notion of vector bundle with extra structure was introduced by R. Hermann. Also W. Greub, R.G. Swan, R. Narasimhan and H. Flanders extended the work of R. Hermann. Later S. Cairns, R. L. Bishop and R. J. Crittenden generalized the properties of Riemannian and pseudo-Riemannian vector bundles with extra structure. A vector bundle is a quadruple $\xi = (E, \pi, B, F)$ where

(a) (E, π, B, F) is a smooth fibre bundle

(b) F and the fibres $F_x = \pi^{-1}(x), x \in B$ are real linear spaces

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(c) there is a coordinate representation $\{(U_\alpha, \psi_\alpha)\}$ such that the maps $\psi_{\alpha,x} : F \to F_x$ are linear isomorphisms.

A neighbourhood U in B is called a trivializing neighbourhood for ξ if there is a diffeomorphism $\psi_U: U \times F \to \pi^{-1} U$ such that $\pi\psi_U(x, y) = x$ $(x \in U, y \in F)$ and the induced maps $\psi_{U,x}: F \to F_x$ are linear isomorphisms (Bishop and Crittenden, 1964). ψ_U is called a trivializing map for ξ. If $\xi = (E, \pi, B, F)$ and $\xi' = (E', \pi', B', F')$ are vector bundles, a bundle map (also called a homomorphism of vector bundles) $\varphi: \xi \to \xi'$is a smooth fibre-preserving map $\varphi: E \to E'$ such that the restrictions $\varphi_x: F_x \to F'_{\psi(x)}, x \in B$ are linear. If $\varphi': \xi' \to \xi''$ is a second bundle map, then $\varphi' \circ \varphi$ is also a bundle map (Greub and Stamm, 1966).

Let ψ, ψ' and ψ'' denote the smooth maps of base manifolds induced by φ, φ' and $\varphi' \circ \varphi$, then $\psi'' = \psi' \circ \psi$. A bundle map $\varphi: \xi \to \eta$ is called an isomorphism if it is a diffeomorphism. The inverse of a bundle isomorphism is obviously again a bundle isomorphism. Inverse bundle isomorphisms induce inverse diffeomorphisms between the base manifolds (Cairns, 1965). Two vector bundles ξ and ξ' are called isomorphic if there is a bundle isomorphism $\varphi: \xi \xrightarrow{\cong} \xi'$. A strong bundle map between two vector bundles with the same base is a bundle map which induces the identity in the base (Kobayashi and Nomizu, 1963). Now let $\varphi: \xi \to \xi'$ be an arbitrary bundle map inducing $\psi: B \to B'$ and choose coordinate representations $\{(U_\alpha, \psi_\alpha)\}$ and $\{(V_i, \chi_i)\}$ for ξ and ξ' respectively. Then smooth maps

$$\varphi_{i\alpha} : \psi^{-1}(V_i) \cap U_\alpha \to L(F; F')$$

are defined by

$$\varphi_{i\alpha}(x) = \chi_{ix'}^{-1} \circ \varphi_x \circ \psi_{\alpha,x}, \quad x' = \psi(x).$$

They are called the mapping transformations for φ corresponding to the given coordinate representations.

Whitney Sum

A vector bundle ξ is called the Whitney sum of the bundles $\xi^\nu(\nu = 1, \cdots, p)$ if there are defined strong bundle maps $i^\nu: \xi^\nu \longrightarrow \xi$ and $\rho^\nu : \xi \longrightarrow \xi^\nu$such that

$$\rho^\nu \circ i^\mu = \begin{cases} 0, & \nu \neq \mu \\ \iota, & \nu = \mu \end{cases} \text{ and } \sum_{\nu=1}^{p} i^\nu \circ \rho^\nu = \iota_\xi.$$

In particular, the fibre F_x in ξ over a point $x \in B$ is then the direct sum of the fibres F_x^ν. In this case ξ is denoted by $\xi^1 \oplus \cdots \oplus \xi^p$. Next, we suppose that $\varphi^\upsilon: \xi^\upsilon \longrightarrow \eta$ are strong bundle maps. Then a strong bundle map $\varphi: \xi \longrightarrow \eta$ is given by

$\varphi = \sum_\nu \varphi^\nu \circ \rho^\nu$. The correspondence $(\varphi^1, \cdots, \varphi^p) \mapsto \varphi$ defines a module isomorphism

$$\oplus_\nu Hom(\xi^\nu; \eta) \xrightarrow{\cong} Hom(\xi^1 \oplus \cdots \oplus \xi^p; \eta).$$

Proposition 1. Let $\varphi: \xi \to \xi'$ be a homomorphism of vector bundles inducing $\psi: B \to B'$ between the base manifolds. Then φ is an isomorphism if and only if

(1) $\psi: B \to B'$ is a diffeomorphism,

(2) each $\varphi_x : F_x \to F'_{\psi(x)} (x \in B)$ is a linear isomorphism.

Proof. If φ is an isomorphism, then (1) and (2) are obvious. Conversely, assume (1) and (2) hold. Then φ is bijective and φ^{-1} restricts to the linear isomorphisms

$$\varphi_x^{-1} : F'_{\psi(x)} \xrightarrow{\cong} F_x.$$

It remains to prove that φ^{-1} is smooth. With the aid of trivializing neighbourhoods for ξ and ξ' we can reduce to the case

$$B = B', \quad E = B \times F, \quad E' = B \times F',$$

where *E*, *E'* are the total manifolds for ξ, ξ' and ψ is the identity map. Then $x \mapsto \varphi_x$ defines a smooth map $\Phi: B \to L(F; F')$ and φ^{-1} is the smooth map given by

$$\varphi^{-1}(x, y') = \left(x, \Phi(x)^{-1}(y')\right), x \in B, y' \in F'$$

which completes the proof.

Proposition 2. The Whitney sum of vector bundles always exists.

Proof. We shall restrict ourselves to the case $p = 2$, since the generalization is obvious. We assign to each $x \in B$ the vector space $F_x^1 \oplus F_x^2$. Let $\{(U_\alpha, \varphi_\alpha^1)\}$ and $\{(U_\alpha, \varphi_\alpha^2)\}$ be coordinate representations for ξ^1, ξ^2 and assign to $x \in U_\alpha$ the linear isomorphism

$$\psi_{\alpha,x} = \varphi_{\alpha,x}^1 \oplus \varphi_{\alpha,x}^2 : F^1 \oplus F^2 \to F_x^1 \oplus F_x^2.$$

Then the construction principle yields a vector bundle

$$\tilde{\xi} = \left(\tilde{E}, \pi, B, F^1 \oplus F^2\right)$$

where $\tilde{E} = F_x^1 \oplus F_x^2$ and obviously π is the projection map. The inclusions

$$F_x^1, F_x^2 \to F_x^1 \oplus F_x^2$$

define strong bundle maps

$$i^1 : \xi^1 \to \tilde{\xi} \text{ and } i^2 : \xi^2 \to \tilde{\xi}.$$

The projections

$$F_x^1 \oplus F_x^2 \to F_x^1 , \; F_x^1 \oplus F_x^2 \to F_x^2$$

define strong bundle maps $\rho^1 : \tilde{\xi} \to \xi^1$ and $\rho^2 : \tilde{\xi} \to \xi^2$. These maps satisfy the required conditions and so $\tilde{\xi}$ is the Whitney sum of ξ^1 and ξ^2. Hence, the Whitney sum of vector bundles always exists.

Vector Bundles with Extra Structure

Let $\xi = (E, \pi, B, F)$ be a vector bundle of rank r with dual bundle ξ^*. Then $\wedge^r \xi^*$ is a vector bundle of rank 1. We say that ξ is orientable if there exists a $\Delta \in Sec \wedge^r \xi^*$ such that $\Delta(x) \neq 0, \; x \in B$. Such a cross-section is called a determinant function in ξ. Clearly $\Delta(x)$ is a determinant function in the vector space F_x.

Theorem 1. A vector bundle $\xi = (E, \pi, B, F)$ is orientable if and only if it admits a coordinate representation $\{(U_\alpha , \varphi_\alpha)\}$ whose coordinate transformations

$$g_{\alpha\beta}(x) = \varphi_{\alpha,x}^{-1} \circ \varphi_{\beta,x}$$

have positive determinant.

Proof. Assume that ξ is orientable and let Δ be a determinant function in ξ. Let $\{(U_\alpha , \psi_\alpha)\}$ be a coordinate representation for ξ such that the U_α are connected. Choose a fixed determinant function Δ_F in F. Since the U_α are connected, for each α, the linear maps

$$\psi_{\alpha,x} : F \to F_x, \qquad x \in U_\alpha$$

either all preserve or all reverse the orientations. Let ρ be an orientation-reversing isomorphism of F and define a coordinate representation $\{(U_\alpha , \varphi_\alpha)\}$ for ξ by

$$\varphi_\alpha(x, y) = \begin{cases} \psi_\alpha(x, y), \text{if } \psi_{\alpha,x} \text{ preserves orientations} \\ \psi_\alpha\big(x, \rho(y)\big), \text{if } \psi_{\alpha,x} \text{ reverses orientations} \end{cases}.$$

Then each $\varphi_{\alpha,x}$ preserves orientations. Hence $\varphi_{\alpha,x}^{-1} \circ \varphi_{\beta,x}$ preserves orientations, that is, $\det\big(\varphi_{\alpha,x}^{-1} \circ \varphi_{\beta,x}\big) > 0$.

Conversely, assume ξ that admits a coordinate representation $\{(U_\alpha , \varphi_\alpha)\}$ such that

$$\det\big(\varphi_{\alpha,x}^{-1} \circ \varphi_{\beta,x}\big) > 0, \qquad x \in U_\alpha \cap U_\beta .$$

Let Δ_F be a determinant function in F and define $\Delta_F \in A^r(\xi_{U_\alpha})(r =$

rank ξ)by

$$\Delta_\alpha(x;\, z_1,\cdots,z_r) = \Delta_F\left(\varphi_{\alpha,x}^{-1}(z_1),\cdots,\varphi_{\alpha,x}^{-1}(z_r)\right)$$

A simple computation shows that

$$\Delta_\alpha(x;\, z_1,\cdots,z_r) = \det\left(\varphi_{\alpha,x}^{-1}\circ\varphi_{\beta,x}\right)\Delta_\beta(x;\, z_1,\cdots,z_r),\, x\in U_\alpha\cap U_\beta,\;\; z_i\in F_x.$$

Now, assume that the cover $\{U_\alpha\}$ of B is locally finite, and let $\{p_\alpha\}$ be a subordinate partition of unity. Define $\Delta\in Sec\wedge^r\xi^*$ by

$$\Delta(x;\, z_1,\cdots,z_r) = \sum_\alpha p_\alpha(x)\,\Delta_\alpha(x;\, z_1,\cdots,z_r),\, x\in B, z_i\in F_x.$$

Since $\sum_\alpha p_\alpha = 1, p_\alpha(x)\geq 0$ and $\det\left(\varphi_{\alpha,x}^{-1}\circ\varphi_{\beta,x}\right) > 0$, it follows that $\Delta(x)\neq 0, x\in B$. So ξ is orientable and the theorem is proved.

Riemannian and Pseudo-Riemannian Vector Bundles

Let $\xi = (E,\pi,B,F)$ be a vector bundle. A pseudo-Riemannian metric in ξ is an element $g\in S^2(\xi)$ such that, for each $x\in B$, the symmetric bilinear form $g(x)$ in F_x is nondegenerate. The pair (ξ, g) is called a pseudo-Riemannian vector bundle.

If the bilinear forms $g(x)$ are positive definite for every $x\in B$, then g is called a Riemannian metric and (ξ, g) is called a Riemannian vector bundle. A cross-section σ in a pseudo-Riemannian vector bundle is called normed if $g\big(x;\sigma(x),\sigma(x)\big) = 1, x\in B$.

Proposition 3. Every vector bundle ξ admits a Riemannian metric.

Proof. If $\xi = B\times F$ is trivial and $\langle,\rangle$ is a Euclidean metric in F, then

$$g(x;\, y_1,y_2) = \langle y_1,y_2\rangle,\;\; x\in B,\qquad y_1, y_2\in F$$

defines a Riemannian metric in ξ.

Now, let ξ be arbitrary and let $\{(U_\alpha,\varphi_\alpha)\}$ be a coordinate representation for ξ such that $\{U_\alpha\}$ is a locally finite open cover of B. Let $\{p_\alpha\}$ be a subordinate partition of unity. Since the restriction ξ_α of ξ to U_α is trivial, there is a Riemannian metric g_α in ξ_α. We define g by $\sum_\alpha p_\alpha g_\alpha$. Then $g(x)$ is a Euclidean metric in F_x; hence g is a Riemannian metric in ξ. Therefore, every vector bundle ξ admits a Riemannian metric.

Definition 1. Let $\xi = (E_\xi, \pi_\xi, B, F)$ and $\eta = (E_\eta, \pi_\eta, B', H)$ be Riemannian bundles and let $\varphi: \xi \to \eta$ be a bundle map. φ is called isometric if the linear maps φ_x are isomorphisms which preserve the inner product.

Theorem 2. Let $\xi = (E, \pi, B, F)$ be a vector bundle with Riemannian metric g. Let $\langle , \rangle$ be a fixed Euclidean inner product in F. Then there is a coordinate representation $\{(U_\alpha, \varphi_\alpha)\}$ for ξ such that the maps

$$\varphi_{\alpha,x} : F \to F_x$$

are isometries.

Proof. It is sufficient to consider the case that $\xi = B \times F$ is trivial. We denote $g(x)$ by $\langle , \rangle_x$ and let F_x denote the vector space F endowed with the inner product $\langle , \rangle_x$. Let $\{e_1, \cdots, e_r\}$ be an orthonormal basis of F with respect to $\langle , \rangle$.

Now let $\{\tau_1(x), \cdots, \tau_r(x)\}$ be the orthonormal basis of F_x obtained from $\{e_1, \cdots, e_r\}$ by the Gram-Schmidt process:

$$\tau_i(x) = \langle \omega_i(x), \omega_i(x) \rangle_x^{-1/2} \omega_i(x),$$

where

$$\omega_i(x) = e_i - \sum_{j=1}^{i-1} \langle e_i, \tau_j(x) \rangle_x \tau_j(x).$$

It follows from this formula that the maps $\tau_i: B \to F$ are smooth. Hence a coordinate representation for ξ is given by (B, ψ), where $\psi : B \times F \to E$ is defined by

$$\psi(x, y) = \left(x, \sum_i \langle e_i, y \rangle \tau_i(x) \right).$$

Moreover, each $\psi_x : E \to F_x$ is an isometry. Hence the proof of the theorem is complete.

Definition 2. Let $\xi = (E, \pi, B, F)$ be a vector bundle with Riemannian metric g. Let $\langle , \rangle$ be a fixed Euclidean inner product in F. Then the coordinate representation $\{(U_\alpha, \varphi_\alpha)\}$ is called a Riemannian coordinate representation for ξ such that the maps $\varphi_{\alpha,x} : F \to F_x$ are isometries.

Proposition 4. If (ξ, g) and (η, h) are Riemannian vector bundles over the same base B and $\varphi \in Hom(\xi;\eta)$ is an isomorphism, then there exists an isometric isomorphism

$$\psi : \xi \xrightarrow{\cong} \eta.$$

Proof. Since h induces a Riemannian metric $\tilde{h}$ on ξ with respect to φ which is an isometry, we may assume that $\eta = \xi$. We define $\alpha \in Hom(\xi;\xi)$ by

$$h(x;\ \alpha_x(z), w) = g(x; z, w), x \in B,\ \ z, w \in F_x.$$

Since $h(x)$ and $g(x)$ are inner products, each $\alpha_x \in S^+(F_x)$. There is a unique $\psi_x \in S^+(F_x)$ which satisfies $\psi_x^2 = \alpha_x$ and which depends smoothly on α_x. Thus the induced bundle map $\psi: \xi \rightarrow \xi$ is a strong isometric isomorphism, which completes the proof.

Lemma 1. Suppose that $\eta = (E_\eta, \pi_\eta, B, H)$ is a subbundle of a Riemannian vector bundle $(\xi = (E, \pi, B, F),\ g)$ and that $\langle\, ,\rangle$ is an inner product for F. Then there exists a Riemannian coordinate representation $\{(U_a, \varphi_a) : a \in B\}$ for ξ such that if ψ_a is the restriction of φ_a to $U_a \times H$, then $\{(U_a, \psi_a)\}$ is a coordinate representation of η.

Proof. For each $a \in B$ we can find a neighbourhood V_a and a basis $\{\sigma_1, \cdots, \sigma_s\}$ $(s = \text{rank}\,\eta)$ of $Sec(\eta_{V_a})$. In particular $\{\sigma_1(a), \cdots, \sigma_s(a)\}$ is a linearly independent set of vectors in F_a and so there are $\sigma_{s+1}, \cdots, \sigma_r \in Sec\,\xi$ such that $\{\sigma_1(a), \cdots, \sigma_r(a)\}$ is a basis for F_a. In view of the continuity of the map

$$x \mapsto \sigma_1(x) \wedge \cdots \wedge \sigma_r(x) \in \textstyle\bigwedge^r F_x,\ \ x \in V_a, r = \text{rank}\ \xi,$$

there exists a neighbourhood U_a of a such that $\{\sigma_1(x), \cdots, \sigma_r(x)\}$ forms a basis of F_x, $x \in U_a$. Applying the Gram-Schmidt process we obtain new cross-sections $\tau_1, \cdots, \tau_r$ in $Sec\,(\xi_{U_a})$ such that $\{\tau_1(x), \cdots, \tau_r(x)\}$ is an orthonormal basis of F_x with respect to $g(x)$. Since $\{\sigma_1(x), \cdots, \sigma_s(x)\}$ is a basis of H_x $(x \in U_a)$, it follows from the construction that $\{\tau_1(x), \cdots, \tau_s(x)\}$ is a basis of H_x.

Now we choose an orthonormal basis $\{e_1, \cdots, e_r\}$ of F such that $\{e_1, \cdots, e_s\}$ is a basis for H. We define maps $\varphi_a : U_a \times F \rightarrow \pi^{-1}U_a$ by

$$\varphi_a\left(x, \sum \lambda_i\, e_i\right) = \sum \lambda_i\, \tau_i(x), \quad x \in U_a.$$

Then $\{(U_a, \varphi_a): a \in B\}$ is the required coordinate representation of ξ, which completes the proof.

Proposition 5. If η is a subbundle of ξ, there is a second subbundle ζ of ξ such that ξ is the Whitney sum of η and ζ.

Proof. We assign to ξ a Riemannian metric g. Choose an inner product $\langle\,,\rangle$ for F and let $\{(U_a, \varphi_a): a \in B\}$ be a coordinate representation for ξ satisfying the conditions of Lemma 1.

To construct the subbundle ζ we use the construction principle. We assign the vector space $H_x^{\perp}$ to $x \in B$,

$$H_x^{\perp} = \{z \in F_x : g(x; z, w) = 0 \text{ when } w \in H_x\}.$$

Since $\varphi_{a,x}: F \to F_x \ (x \in U_a)$ is an isometry which carries H to H_x, it restricts to a linear isomorphism $\psi_{a,x}: H^{\perp} \xrightarrow{\cong} H_x^{\perp}$.

The induced maps $U_a \cap U_b \to L(H^{\perp}; H^{\perp})$ given by $x \mapsto \psi_{b,x}^{-1} \circ \psi_{a,x}$ are smooth. Thus we obtain a vector bundle $\zeta = (E_\zeta, \pi_\zeta, B, H^{\perp})$, where $E_\zeta = \bigcup_x H_x^{\perp}$. Evidently ζ is a subbundle of ξ. The inclusions $i: \eta \to \xi$, $j: \zeta \to \xi$ extend to a strong bundle map $\eta \oplus \zeta \to \xi$. Since $F_x = H_x \oplus H_x^{\perp}$, this map restricts to isomorphisms in each fibre; hence it is an isomorphism, which completes the proof.

Oriented Riemannian Bundles and Complex Vector Bundles

Assume that g is a Riemannian metric in an oriented vector bundle $\xi = (E, \pi, B, F)$ of rank r. Let ξ^* be any dual bundle. Then the induced isomorphism $\xi \cong \xi^*$ induces a Riemannian metric in ξ^* and hence in $\wedge^r \xi^*$. There is a unique normed cross-section $\Delta \in Sec \wedge^r \xi^*$ which is positive with respect to the orientation of ξ. It is called the positive normed determinant function in ξ. For each $x \in B$, $\Delta(x)$ is the positive normed determinant function in F_x. Let $\xi = (E, \pi, B, F)$ be a Riemannian vector bundle of rank r and consider the rank 1 bundle $\wedge^r \xi$. Let S_x denote the unit sphere of the one-dimensional Euclidean space $\wedge^r F_x (x \in B)$, then there is a smooth bundle $\tilde{\mathcal{B}} = (\tilde{B}, \rho, B, S^0)$ such that $\rho^{-1} = S_x$.

Proposition 6. If B is connected, then $\tilde{B}$ is connected if and only if ξ is not orientable.

Proof. Since ρ preserves open and closed sets, so it maps each component of $\tilde{B}$ on to the connected manifold B. Since $\rho^{-1}(x)(x \in B)$ consists of two points, there are two possibilities:

(a) $\tilde{B}$ is connected

(b) $\tilde{B}$ has two components $\tilde{B}_1, \tilde{B}_2$ and ρ restricts to diffeomorphisms $\rho_i: \tilde{B}_i \xrightarrow{\cong} B$.

If $\tilde{B}$ is not connected $\rho^{-1}: B \to \tilde{B}_1$ may be interpreted as a cross-section with no zeros in $\wedge^r \xi$; hence ξ is orientable.

Conversely, suppose that ξ is orientable. Choose orientations in ξ and in F, and choose ψ_α, so that each $\psi_{\alpha,x}$ is orientation preserving. Then

$$\varphi_{\alpha,x} = \varphi_{\beta,x} : S^0 \to S_x, x \in U_\alpha \cap U_\beta.$$

Thus φ_α defines a diffeomorphism

$$\varphi: B \times S^0 \xrightarrow{\cong} \tilde{B}.$$

In particular, $\tilde{B}$ is not connected. Hence, if B is connected, then $\tilde{B}$ is connected if and only if ξ is not orientable.

Definition 3. A complex vector bundle is a quadruple $\xi = (E, \pi, B, F)$ where

(a) (E, π, B, F) is a smooth fibre bundle,

(b) F and the fibres F_x $(x \in B)$ are complex linear spaces,

(c) there is a coordinate representation $\{(U_\alpha, \psi_\alpha)\}$ for ξ such that the maps $\psi_{\alpha,x}: F \to F_x$ are complex linear isomorphisms.

Let $\xi = (E, \pi, B, F)$ be a complex vector bundle of rank r. Let $F_{\mathbb{R}}$ be the $2r$-dimensional real vector space underlying F and let $i: F_{\mathbb{R}} \to F_{\mathbb{R}}$ be multiplication by $i \in \mathbb{C}$. Let $\xi_{\mathbb{R}} = (E, \pi, B, F_{\mathbb{R}})$ be the real vector bundle obtained by forgetting the complex structure and let $i_\xi \in L(\xi_{\mathbb{R}}; \xi_{\mathbb{R}})$ be the strong bundle isomorphism which restricts to multiplication by i in each $(F_x)_{\mathbb{R}} = (F_{\mathbb{R}})_x$. Then, if $\{(U_\alpha, \psi_\alpha)\}$ is a coordinate representation for ξ, we have $\psi_{\alpha,x} \circ i = i_\xi(x) \circ \psi_{\alpha,x}$, $x \in U_\alpha$. i_ξ is called the complex structure of ξ.

Proposition 7. Let $\eta = (E, \pi, B, F_{\mathbb{R}})$ be a real vector bundle of rank $2r$. Let $\gamma \in L_\eta$ be a strong bundle map such that $\gamma^2 = -\iota$. Let F be a complex space with underlying real space $F_{\mathbb{R}}$. Then η is the underlying real vector bundle of a complex bundle $\xi = (E, \pi, B, F)$ with complex structure γ.

Proof. We must find a coordinate representation $\{(U_\alpha, \psi_\alpha)\}$ for η such that $\psi_{\alpha,x} \circ i = i_\xi(x) \circ \psi_{\alpha,x}$, $x \in U_\alpha$.

Let $a \in B$ be arbitrary and choose a basis for $(F_{\mathbb{R}})_a$ of the form $z_z, \cdots, z_r, \gamma_a z_1, \cdots, \gamma_a z_r$. There are $\sigma_\nu \in Sec\,\eta$ such that

$$\sigma_\nu(a) = z_\nu (\nu = 1, \cdots, r).$$

By the continuity of the map

$$x \mapsto \sigma_1(x) \wedge \cdots \wedge \sigma_r(x) \wedge (\gamma_* \sigma_1)(x) \wedge \cdots \wedge (\gamma_* \sigma_r)(x),$$

there is a neighbourhood U of a such that $\{\tau_\nu = \sigma_\nu|_U,\ \bar{\tau}_\nu = (\gamma_* \sigma_\nu)|_U,\ \nu = 1, \cdots, r\}$ form a basis for $Sec\ (\eta|_U)$.

Let $\{e_1, \cdots, e_r, i(e_1), \cdots, i(e_r)\}$ be a basis for $F_{\mathbb{R}}$ and we define $\varphi: U \times F_{\mathbb{R}} \to \pi^{-1} U$ by

$$\varphi(x, e_\nu) = \tau_\nu(x),$$

$$\varphi\big(x, i(e_\nu)\big) = \bar{\tau}_\nu(x), \quad \nu = 1, \cdots, r.$$

(U, φ) is a trivializing chart of ξ and $\varphi_x \circ i = \gamma_x \circ \varphi_x, x \in U$. Since U is a neighbourhood of an arbitrary point $a \in B$, the proposition is proved.

Conclusion

The results are unchanged if we replace $\mathbb{R}$ by $\mathbb{C}$ and real vector bundles by complex vector bundles. In particular we have the notion of complex bundle maps (the fibres being complex linear), the module of complex *p*-linear mappings. Also a Hermitian metric can be introduced in every complex vector bundle. If (ξ, g) is a Hermitian complex vector bundle, there exists a coordinate representation $\{(U_\alpha, \varphi_\alpha)\}$ of ξ such that the mappings $\varphi_{\alpha,x}: F \longrightarrow F_x$ are Hermitian isometries.

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